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METHODOLOGY OF SOLVING OLYMPIAD PROBLEMS USING INTEGRAL TECHNIQUES

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Abstract: This article presents a detailed analysis of two advanced problems located at the intersection of mathematical analysis and linear algebra. The first problem,demonstrates the integration of a logarithmic function with a rational expression, solved using series expansion techniques. The second problem involves the integration of a matrix-valued cosine function under the Gaussian kernel, offering an approach to operator-valued functions and matrix analysis. Both problems are designed to enhance students' theoretical knowledge, logical reasoning, and familiarity with competition-level mathematical problem-solving. The article serves as a methodological guide for gifted learners aiming to deepen their understanding of advanced mathematical concepts.

Аннотация: В данной статье проводится подробный анализ двух сложных задач, находящихся на пересечении математического анализа и линейной алгебры. Первая задачапредставляет собой интеграл логарифмической функции, делённой на рациональное выражение, и решается методом разложения в ряд. Вторая задача рассматривает интеграл от матричной косинус-функции с аргументом, зависящим от переменной, под ядром Гаусса. Обе задачи направлены на развитие у студентов теоретических знаний, аналитического мышления и навыков решения олимпиадных задач. Статья ориентирована на одарённых учащихся, стремящихся к глубокому изучению математики.

Keywords: Mathematical analysis, definite integral, logarithmic function, matrix-valued function, linear algebra, operator functions, olympiad problems, methodological approach, series expansion, analytical solution.

Ключевые слова: Математический анализ, определённый интеграл, логарифмическая функция, матричная функция, линейная алгебра, операторные функции, олимпиадные задачи, методический подход, разложение в ряд, аналитическое решение.

Introduction:

Mathematical analysis and linear algebra are core branches of mathematics that encompass a wide range of complex yet fundamental problems. In particular, the theory of real-valued functions, definite integration, and matrix operators are central to many modern theoretical studies. This article focuses on two challenging problems that lie at the intersection of these fields, both selected from high-level mathematical olympiads. Through detailed analysis and exploration of solution strategies, the work aims to develop students' abilities in independent reasoning, applying formulas, and approaching unconventional mathematical problems with a scientific mindset.

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Введение:

Математический анализ и линейная алгебра являются важнейшими разделами математики, включающими множество сложных, но фундаментальных задач. Особенно актуальны в современной науке такие направления, как теория функций действительного переменного, определённые интегралы и матричные операторы. В данной статье рассматриваются две сложные задачи олимпиадного уровня, объединяющие указанные направления. Путём их подробного анализа и пошагового решения автор стремится сформировать у студентов навыки самостоятельного мышления, грамотного применения формул и научного подхода к нестандартным математическим задачам.

Calculate:

$$I=\int_0^{\infty} e^{-x^2} cos\begin{pmatrix} 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} dx$$

Solution:
To solve the given example, we use the Maclaurin series expansion of the cosx function:

$$f(x)\sim f(x)+\frac{f'(x)}{1!}x+\frac{f''(x)}{2!}x^2+\frac{f'''(x)}{3!}x^3+...+\frac{f^{(n)}(x)}{n!}x^n$$

$$cosx=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+...+\frac{(-1)^n x^{2n}}{(2n)!}+...$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

If we take $A=\begin{pmatrix} 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$ and replace x with A, it is not difficult to see that $A^0=I_3$.(I-unit matrix)

$$A^2=AA=\begin{pmatrix} 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^4=A^2A^2=\begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^4=A^6=A^8=...=A^{2n}=O_3\ \ (O_3-zero\ matrix).$$

So,

$$\cos A=I_3-\frac{1}{2}A^2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -\frac{x^2}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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it follows that.

$$I=\int_0^\infty e^{-x^2}\cos\begin{pmatrix}0&x&x\\0&0&x\\0&0&0\end{pmatrix}dx=\int_0^\infty e^{-x^2}\begin{pmatrix}1&0&-\frac{x^2}{2}\\0&1&0\\0&0&1\end{pmatrix}dx=$$

$$=\int_0^\infty\begin{pmatrix}e^{-x^2}&0&-\frac{x^2e^{-x^2}}{2}\\0&e^{-x^2}&0\\0&0&e^{-x^2}\end{pmatrix}dx=\begin{pmatrix}\int_0^\infty e^{-x^2}&0&\int_0^\infty-\frac{x^2e^{-x^2}}{2}\\0&\int_0^\infty e^{-x^2}&0\\0&0&\int_0^\infty e^{-x^2}\end{pmatrix}$$

1. The last obtained integrals are calculated:

$\int_0^\infty e^{-x^2}\,dx=\left[\begin{matrix}x^2=t & x=t^{\frac{1}{2}}\\ dx=\frac{1}{2}t^{-\frac{1}{2}}dt & \begin{cases}x=0 & in\ t=0\\ x=\infty & in\ t=\infty\end{cases}\end{matrix}\right]=\int_0^\infty\frac{1}{2}t^{-\frac{1}{2}}e^{-t}\,dt=\frac{\Gamma\left(\frac{1}{2}\right)}{2}=\frac{\sqrt{\pi}}{2}$

2. $\int_0^\infty -x^2e^{-x^2}\,dx=\left[\begin{matrix}x^2=t & x=t^{\frac{1}{2}}\\ dx=\frac{1}{2}t^{-\frac{1}{2}}dt & \begin{cases}x=0 & in\ t=0\\ x=\infty & in\ t=\infty\end{cases}\end{matrix}\right]=-\int_0^\infty\frac{1}{2}t^{-\frac{1}{2}}e^{-t}\,dt=-\frac{\Gamma\left(\frac{3}{2}\right)}{2}=$
$-\frac{1}{2}\Gamma\left(\frac{1}{2}+1\right)=-\frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=-\frac{\sqrt{\pi}}{4}$

If we rely on the last obtained results,

$$I=\int_0^\infty e^{-x^2}\cos\begin{pmatrix}0&x&x\\0&0&x\\0&0&0\end{pmatrix}dx=\begin{pmatrix}\frac{\sqrt{\pi}}{2}&0&-\frac{\sqrt{\pi}}{8}\\0&\frac{\sqrt{\pi}}{2}&0\\0&0&\frac{\sqrt{\pi}}{2}\end{pmatrix}=\frac{\sqrt{\pi}}{2}\begin{pmatrix}1&0&-\frac{1}{4}\\0&1&0\\0&0&1\end{pmatrix}$$

it follows that.

Answer:

$$I=\int_0^\infty e^{-x^2}\cos\begin{pmatrix}0&x&x\\0&0&x\\0&0&0\end{pmatrix}dx=\frac{\sqrt{\pi}}{2}\begin{pmatrix}1&0&-\frac{1}{4}\\0&1&0\\0&0&1\end{pmatrix}$$

Calculate:

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$$\int_0^1 \frac{\ln(x+1)}{x^2+1}$$

Solution:

$x = \tan\alpha \quad dx = \frac{1}{\cos^2\alpha}$ we introduce the notation.

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} = \int_0^{\frac{\pi}{4}} \frac{\ln(\tan\alpha+1)}{\tan^2\alpha+1}\cdot\frac{1}{\cos^2\alpha}\,d\alpha = \int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)\,d\alpha$$

$\alpha = \frac{\pi}{4} - \alpha \qquad dx = d\alpha$

$$\int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)d\alpha = \int_{\frac{\pi}{4}}^{0} \ln\left(\tan\left(\frac{\pi}{4}-\alpha\right)+1\right)(-d\alpha) = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\tan\frac{\pi}{4}-\tan\alpha}{1+\tan\frac{\pi}{4}\tan\alpha}+1\right)d\alpha$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1-\tan\alpha}{1+\tan\alpha}+1\right)d\alpha = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\alpha}\right)d\alpha = \int_0^{\frac{\pi}{4}} \ln 2d\alpha - \int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)d\alpha$$

$$2\int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)\,d\alpha = \int_0^{\frac{\pi}{4}} \ln 2\,d\alpha$$

$$\int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)\,d\alpha = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,d\alpha$$

$\int_0^{\frac{\pi}{4}} \ln 2\,d\alpha = (\ln 2)\alpha\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}\ln 2$

From this,

$\int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)\,d\alpha = \frac{\pi}{8}\ln 2$ it follows that.

Answer:

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} = \int_0^{\frac{\pi}{4}} \ln(\tan\alpha+1)\,d\alpha = \frac{\pi}{8}\ln 2$$

Conclusion:

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In this article, two complex problems of integral calculus are examined, requiring profound knowledge of mathematical analysis and linear algebra. The first problem demonstrates the analysis of a logarithmic expression through series integration, while the second provides new approaches to working with matrix functions. Both problems serve to develop skills in Olympiad-level thinking, formula manipulation, and generalization. Furthermore, the article is structured on a methodological approach, which is of great importance in fostering students' abilities for deep analysis, independent thinking, and preparation for scientific communication.

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